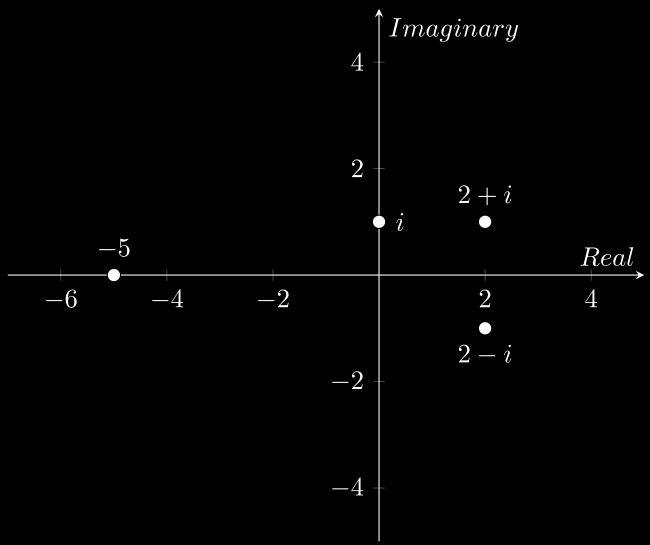
1 Complex notes (Boyle's notes) odds, 10.1 WS

1.1 Problem 1

1. Plot the following points in the complex plane: $i, 2-i, 2+i, -5$.

Begin by drawing a real and imaginary axis. Then i has coordinates $(0,1)$, $2-i$ has coordinates $(2,-1)$, $2+i$ has coordinates $(2,1)$, and -5 has coordinates $(-5,0)$:



1.2 Problem 3

3. (Multiplicative inverse) Let $z = a + ib$ be a nonzero complex number (so at least one of the real numbers a, b is nonzero). Then $1/z$ is the number such that $(z)(1/z) = 1$, and there is only one such number.

(i) Show that $1/z = \frac{a-ib}{a^2+b^2}$ [Hint: Just show that multiplying the right hand side by $a + ib$ produces the number 1; then the right hand side must be a correct formula for $1/z$.]

(ii) Compute real numbers a, b such that $1/(2 + 3i) = a + ib$.

(iii) Compute real numbers a, b such that $(1 - 2i)/(2 + 3i) = a + ib$.

(iv) If the polar form of z is $Re^{i\theta}$, then what is the polar form of $1/z$?

(i) Show that $1/z = \frac{a-ib}{a^2+b^2}$

Note that $a + ib \cdot \frac{a-ib}{a^2+b^2} = \frac{a^2+aib-aib-(ib)^2}{a^2+b^2} = \frac{a^2-(-b^2)}{a^2+b^2} = 1$. Thus $z \cdot 1/z = 1$, so $\frac{a-ib}{a^2+b^2}$ must be the correct formula for $1/z$.

(ii) Compute real numbers a, b such that $1/(2 + 3i) = a + ib$.

By the formula, $1/(2 + 3i) = \frac{2-3i}{2^2+3^2} = \frac{2}{13} - \frac{3}{13}i$.

(iii) Compute real numbers a, b such that $(1-2i)/(2+3i) = a+ib$.

By part 2,

$$\frac{1-2i}{2+3i} = (1-2i)(\frac{2}{13} - \frac{3}{13}i) = (\frac{2}{13} - \frac{3}{13}i) - 2i(\frac{2}{13} - \frac{3}{13}i) = \frac{2}{13} - \frac{3}{13}i - \frac{4}{13}i - \frac{6}{13} = -\frac{4}{13} - \frac{7}{13}i$$

Thus $a = -4/13$, $b = -7/13$.

(iv) If the polar form of z is $Re^{i\theta}$, then what is the polar form of $1/z$?

Note $1/z = 1/(Re^{i\theta}) = \frac{1}{R} \cdot \frac{1}{e^{i\theta}} = \frac{1}{R}e^{-i\theta}$. This is thus the polar form of $1/z$ with $r = 1/R$ and angle $-\theta$.

1.3 Problem 5

5. Now consider a complex number z written in various forms: $z = x + iy = e^{a+ib} = Re^{i\theta}$, where x, y, a, b, R and θ are real numbers. (i) Give formulas using x and y for R and $\tan(\theta)$. For which z are the formulas valid?

(ii) Give formulas for R and in terms of x and y.

(iii) Compute the polar form of e^{2-3i}.

(i) Give formulas using x and y for R and $\tan(\theta)$. For which z are the formulas valid?

We have $R = \sqrt{x^2+y^2}$ and $\tan(\theta) = \frac{y}{x}$ for all z with $x \neq 0$.

(ii) Give formulas for R and in terms of x and y.

We have $R = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(\frac{y}{x})$.

(iii) Compute the polar form of e^{2-3i}.

The polar form of $e^{2-3i} = e^2 \cdot e^{-3i}$ is $Re^{i\theta}$ with $R = e^2$, $\theta = -3$.

1.4 Problem 7

7. (Complex conjugates) Let $z = a + ib$; then the complex conjugate $\bar{z}$ is defined to be $\bar{z} = a - ib$.

(i) How are the locations of z and $\bar{z}$ in the complex plane related?

(ii) Check that $z\bar{z} = |z|^2 = a^2 + b^2$.

(iii) Show that if z is nonzero, then $1/z = \bar{z}/|z|^2$. (Multiply z by this expression and check that the product is 1.)

(iv) Use the formula in (iii) to find $1/z$ if $z = 2 + 3i$.

(i) How are the locations of z and $\bar{z}$ in the complex plane related?

The angle of z is made negative so $\bar{z}$ is a reflection of z through the Real axis; an example is

$$e^{i\pi/4} \to e^{i\bar{\pi}/4} = e^{-i\pi/4}$$

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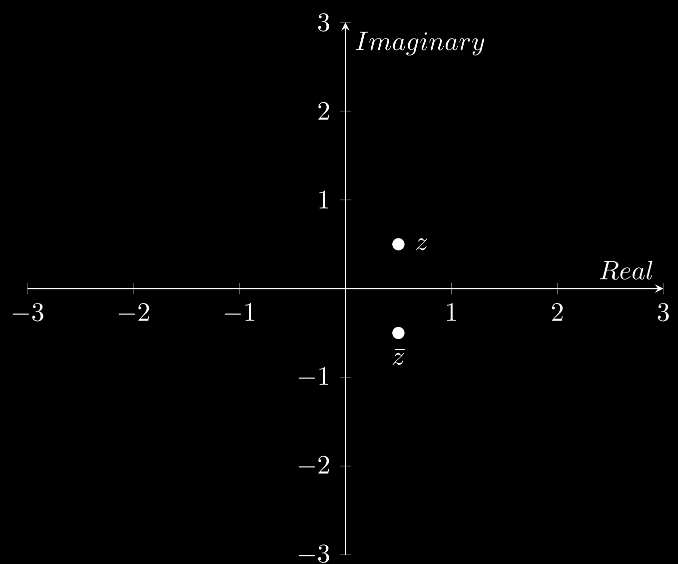


(ii) Check that $z\bar{z} = |z|^2 = a^2 + b^2$.

Note $z\bar{z} = (a + bi)(a - bi) = a^2 + abi - abi + (bi)^2 = a^2 + b^2 = (\sqrt{a^2 + b^2})^2 = |z|^2$.

(iii) Show that if z is nonzero, then $1/z = (\bar{z})/(|z|^2)$. (Multiply z by this expression and check that the product is 1.)

By our formula from problem 3, $1/z = \frac{a-ib}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$.

(iv) Use the formula in (iii) to find $1/z$ if $z = 2 + 3i$.

We found this in problem 3: $1/(2 + 3i) = 2/13 - i(3/13)$.

1.5 Problem 9

9. From the last problem, it follows that if $p(z)$ is a polynomial with real coefficients and w is a complex number and $p(w) = 0$, then also $p(\bar{w}) = 0$. (i) Check that any polynomial of the form $q(z) = (z - w)(z - \bar{w})$ is a polynomial with real coefficients.

(ii) (Real Factorization Theorem) Deduce using the Factorization The- orem that any nonconstant polynomial with only real coefficients can be factored as a product of polynomials with only real coefficients and with degree one or two.

(i) Check that any polynomial of the form $q(z) = (z - w)(z - \bar{w})$ is a polynomial with real coefficients.

Well $q(z) = (z-w)(z-\bar{w}) = z^2 - wz - \bar{w}z + w\bar{w} = z^2 - (a+bi+(a-bi))z + a^2 + b^2 = z^2 - (2a)z + (a^2 + b^2)$, so $q(z)$ is a polynomial with real coefficients.

(ii) (Real Factorization Theorem) Deduce using the Factorization The- orem that any nonconstant polynomial with only real coefficients can be factored as a product of polynomials with only real coefficients and with degree one or two.

Well, the factorization theorem (from Boyle's notes) states that if $p(z)$ is a polynomial of degree $n \geq 1$, with real (or complex) coefficients, say $p(z) = c_n z^n + c_{n-1} z^{n-1} + ... + c_1 z + c_0$, then p can be factored as a product of linear terms $p(z) = c_n(z - z_1)(z - z_2) \cdots (z - z_n)$ where the numbers $z_1, z_2, ...z_n$ are the roots of $p(z)$ (possibly some roots appear more than once).

Thus if $p(x)$ is any nonconstant polynomial with only real coefficients, and z is a complex root (with nonzero imaginary part), then $z = z_i$ for some i, and by part (i), $\bar{z}$ is also a root, so $\bar{z} = z_j$ for some $j \neq i$. Then $(x - z)(x - \bar{z})$, a polynomial with only real coefficients of degree 2 polynomial, in the factorization.

If z is a real root (so 0 imaginary part) then by the factor theorem $(x - z)$ appears. Thus we can split $p(x) = \prod_{z \text{ complex root of p,Im}(z)>0} (x - z)(x - \bar{z}) \cdot \prod_{z \text{ real root of p}} (x - z)$ into degree 1 or 2 polynomials with only real coefficients (by part (i) $(x - z)(x - \bar{z})$ has only real coefficients).

1.6 Problem 11

11. (Roots of unity) Let n be a positive integer. The complex numbers has its nth power equal to 1. Likewise, if k is a nonnegative integer in the set $0, 1, ..., n - 1$, then $e^{2\pi ik/n}$ also has its nth power equal to 1. Such a number is called an nth root of unity. These numbers can be drawn on the unit circle in the complex plane.

(i) Draw all the fourth roots of unity on the unit circle.

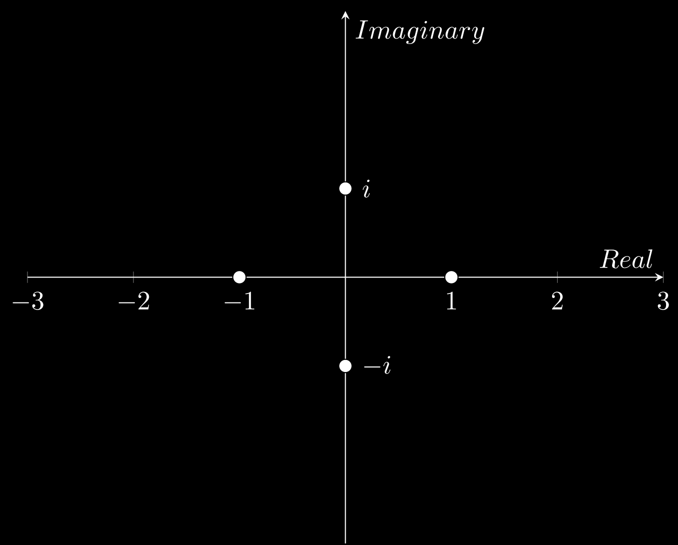
(ii) Draw (in another picture) all the eighth roots of unity.

(i) Draw all the fourth roots of unity on the unit circle.

The fourth roots of unity are

$$e^0 = 1, e^{2\pi i/4} = e^{(\pi/2)i} = i, e^{2\pi i \cdot 2/4} = e^{\pi i} = -1, e^{2\pi i \cdot 3/4} = e^{(3\pi/2)i} = -i$$

Thus:

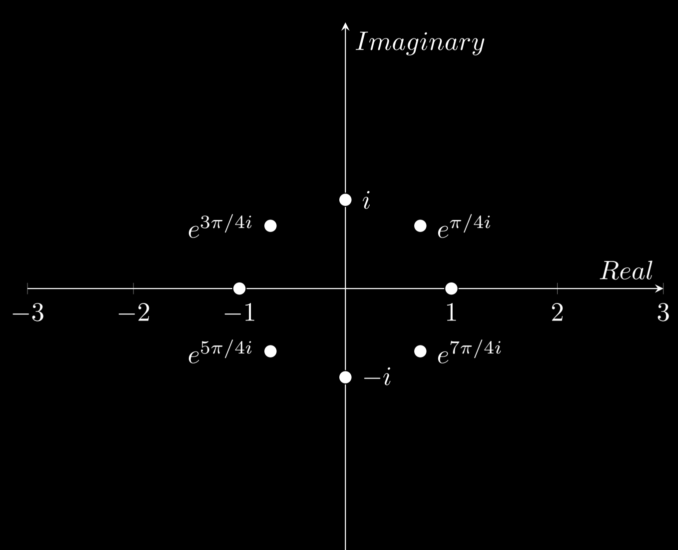


(ii) Draw (in another picture) all the eighth roots of unity.

The eighth roots of unity satisfy $z^8 = 1$, so in particular any fourth root satisfies this equation since $z^4 = 1$ means $(z^4)^2 = z^8 = 1$. Here are all the eighth roots of unity:

k=0	1
k=1	$e^{2\pi/8i} = e^{\pi/4i}$
k=2	i
k=3	$e^{3\pi/4i}$
k=4	-1
k=5	$e^{5\pi/4i}$
k=6	-i
k=7	$e^{7\pi/4i}$

Note $e^{i\pi/4} = \cos(\pi/4) + i\sin(\pi/4) = \sqrt{2}/2 + i\sqrt{2}/2 \approx 0.707 + i0.707$, etc.:



1.7 Problem 13

13. If n is a positive integer and M is a positive real number, then the equation $z^n = M$ has exactly the following n solutions: $M^{1/n}e^{2\pi ki/n}$, $k = 0, 1, 2, ..., n-1$.

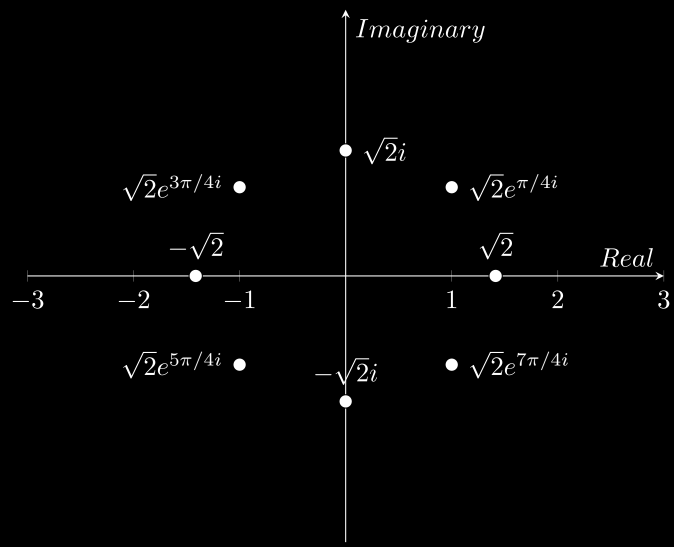
Find all solutions of the equation $z^8 = 16$, and plot these solutions in the complex plane.

These will be almost the same solutions as in Problem 11, just with adjusted magnitude $M^{1/n} = \sqrt[8]{16} = (2^4)^{1/8} = \sqrt{2}$:

Here are all the solutions of unity:

k=0	$\sqrt{2}$
k=1	$\sqrt{2}e^{2\pi/8i} = e^{\pi/4i}$
k=2	$\sqrt{2}i$
k=3	$\sqrt{2}e^{3\pi/4i}$
k=4	-$\sqrt{2}$
k=5	$\sqrt{2}e^{5\pi/4i}$
k=6	-$\sqrt{2}i$
k=7	$\sqrt{2}e^{7\pi/4i}$

Note $e^{i\pi/4} = \cos(\pi/4) + i\sin(\pi/4) = \sqrt{2}/2 + i\sqrt{2}/2$ so $\sqrt{2}e^{i\pi/4} = 1 + 1i$, etc.:



1.8 Problem 15

15. (DeMoivre) To understand why $e^{iz} = \cos(z) + i\sin(z)$, compute by hand the first eight terms of these series, and compare.

The series involved are $e^{iz} = \sum_{n\geq 0} \frac{1}{n!}(iz)^n$, $\cos(z) = \sum_{n\geq 0} \frac{(-1)^n}{(2n)!}z^{2n}$, and $\sin(z) = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!}z^{2n+1}$.

The first eight terms of $\cos(z) + i\sin(z)$ are $1 + iz - z^2/2 - iz^3/3! + z^4/4! + iz^5/5! - z^6/6! - iz^7/7!$

The first eight terms of e^{iz} are

$$1+(iz)+(iz)^2/2+(iz)^3/3!+(iz)^4/4!+(iz)^5/5!+(iz)^6/6!+(iz)^7/7! = 1+iz-z^2/2-iz^3/3!+z^4/4!+iz^5/5!-z^6/6!-iz^7/7!$$

so the first 8 terms agree.

2 WS 10.1

2.1 Problem 1

1. Consider the point $P = (x, y)$ with rectangular coordinates $(1, -\sqrt{3})$. Let (r, θ) be polar coordinates of P , with $r > 0$ and $0 \leq \theta < 2\pi$.

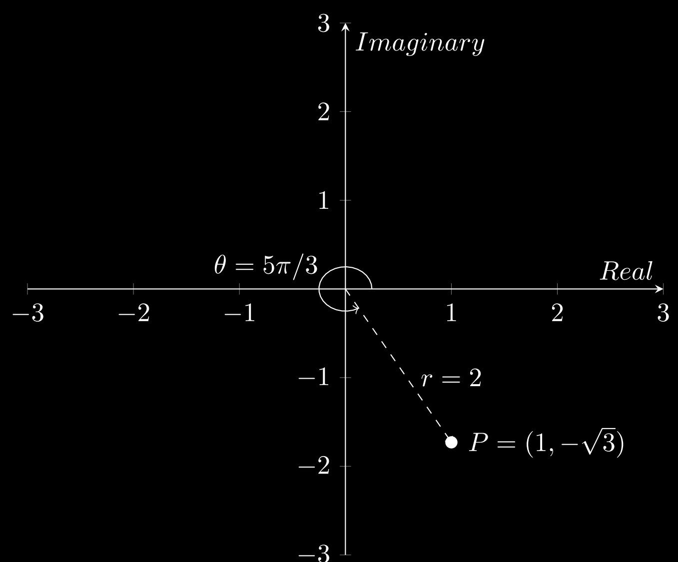
(a) Draw P in the usual Cartesian plane. Then find the values of r and θ, giving reasons.

(b) For the values of r and θ found in part (a), draw on the same large graph (different from the graph in part (a)) the points with the following polar coordinates and label each:

(r,-θ), (r,$\pi - \theta$), (r,$\pi + \theta$), (-r, θ), (-r, $\pi - \theta$), (-r,$\pi/2 - \theta$)

(a) Draw P in the usual Cartesian plane. Then find the values of r and θ, giving reasons.

Since $P = (1, -\sqrt{3})$, we find the distance from the origin is $r = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{4} = 2$, with angle $\theta = \tan^{-1}(-\sqrt{3}/1) = -\pi/3$. Since we want $0 \leq \theta < 2\pi$, add 2π to $-\pi/3$, obtaining $5\pi/3$. Thus the polar form is $(2, 5\pi/3)$. In the Cartesian plane P is:



(b) For the values of r and θ found in part (a), draw on the same large graph (different from the graph in part (a)) the points with the following polar coordinates and label each:

(r,-θ), (r,$\pi - \theta$), (r,$\pi + \theta$), (-r, θ), (-r, $\pi - \theta$), (-r,$\pi/2 - \theta$)

Note $(r, \theta) = (2, 5\pi/3)$, so

$$(r, -\theta) = (2, -5\pi/3) = (2, \pi/3),$$

$$(r, \pi - \theta) = (2, \pi - 5\pi/3) = (2, -2\pi/3) = (2, 4\pi/3),$$

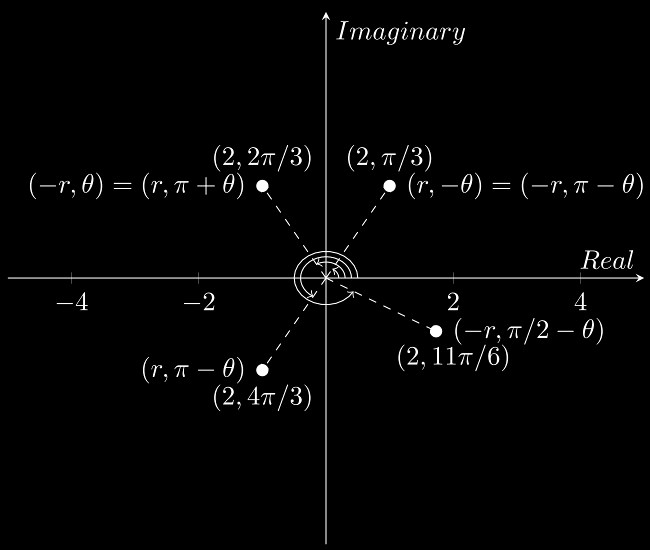
$$(r, \pi + \theta) = (2, \pi + 5\pi/3) = (2, 8\pi/3) = (2, 2\pi/3),$$
$$(-r, \theta) = (-2, 5\pi/3) = (2, 5\pi/3(+\pi)) = (2, 8\pi/3) = (2, 2\pi/3),$$
$$(-r, \pi - \theta) = (r, \pi - \theta(+\pi)) = (r, 2\pi - \theta) = (r, -\theta)$$

and finally

$$(-r, \pi/2-\theta) = (2, \pi/2-\theta(+\pi)) = (2, \pi/2-5\pi/3+\pi) = (2, 3\pi-10\pi)/6(+\pi) = (2, -7\pi/6(+\pi)) = (2, -\pi/6) = (2, 11\pi/3),$$

. Then the points we will graph are

$$\begin{array}{ccc} (\text{r},-\theta) = (2, \pi/3), & (\text{r},\pi - \theta) = (2, 4\pi/3), & (\text{r},\pi + \theta) = (2, 2\pi/3), \\ (\text{-r},\theta) = (2, 2\pi/3), & (\text{-r},\pi - \theta) = (2, \pi/3), & (\text{-r},\pi/2 - \theta) = (2, 11\pi/6) \end{array}$$



2.2 Problem 2

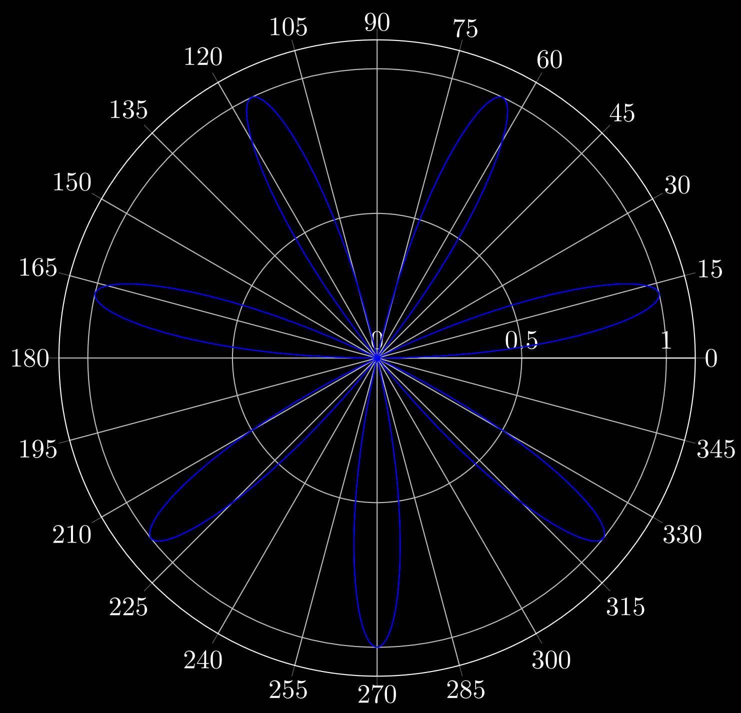
2. (a) Plot the graphs of $r = \sin(7\theta)$ and $r = \sin(4\theta)$, for $0 \le \theta \le 2\pi$. Which rose is traced out twice, and which rose is traced out once as θ increases from 0 to 2π.

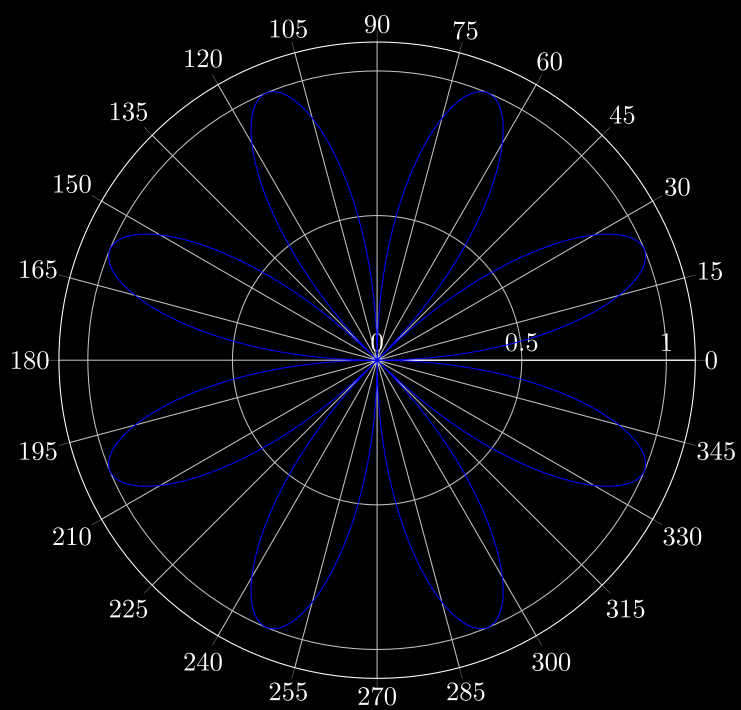
(b) Let n be a positive integer, and consider the graph of $r = \sin(n\theta)$. Determine the number of leaves when n is odd, and the number of leaves when n is even. (The reason there is a difference between even n and odd n is interesting. Can you provide the reason?)

(c) Find a polar equation for a 12-leaved rose.

(a) Plot the graphs of $r = \sin(7\theta)$ and $r = \sin(4\theta)$, for $0 \le \theta \le 2\pi$. Which rose is traced out twice, and which rose is traced out once as θ increases from 0 to 2π.

To plot $r = \sin(7\theta)$ plot significant points for the function, such as $0, (\pi/2)/7, \pi/7$ etc. and keep track of the direction things are travelling.





The rose $r = \sin(7x)$ is traced out twice while the rose $r = \sin(4x)$ is traced out once: (note $(r, \theta) = (1, (\pi/2)/7) = (-1, \pi/14(+\pi)) = (-1, 15\pi/14)$ and $\sin(7 \cdot (15\pi/14)) = \sin(15\pi/2) = \sin(3\pi/2) = -1$, so when θ increases to $15\pi/14$ we return to the point $(1, \pi/14)$. This happens in general: thus we trace out the each point twice in the graph of $\sin(7x)$.

(b) Let n be a positive integer, and consider the graph of $r = \sin(n\theta)$. Determine the number of leaves when n is odd, and the number of leaves when n is even. (The reason there is a difference between even n and odd n is interesting. Can you provide the reason?)

The number of leaves is n when n is odd, and the number of leaves when n is even is $2n$. The reason for the difference is that $\sin(n(\theta+\pi)) = \sin(n\theta+n\pi) = \sin(n\theta)$ when n is even, while $\sin(n\theta+n\pi) = -\sin(n\theta)$ when n is odd.

Thus if $r = \sin(n\theta)$, when n is even, the points $(\sin(n\theta), \theta)$ and $(\sin(n(\theta+\pi)), \theta+\pi)$ are distinct on the graph for each θ, so the graph from $[0, \pi]$ is repeated distinctly on $[\pi, 2\pi]$ (compare this with the graph above).

When n is odd, the points $(\sin(n\theta), \theta)$ and

$$(\sin(n(\theta+\pi)), \theta+\pi) = (-\sin(n\theta), \theta+\pi) = (\sin(n\theta), \theta+\pi(+\pi)) = (\sin(n\theta), \theta)$$

are the same, so the graph on $[\pi, 2\pi]$ retraces the graph from $[0, \pi]$.

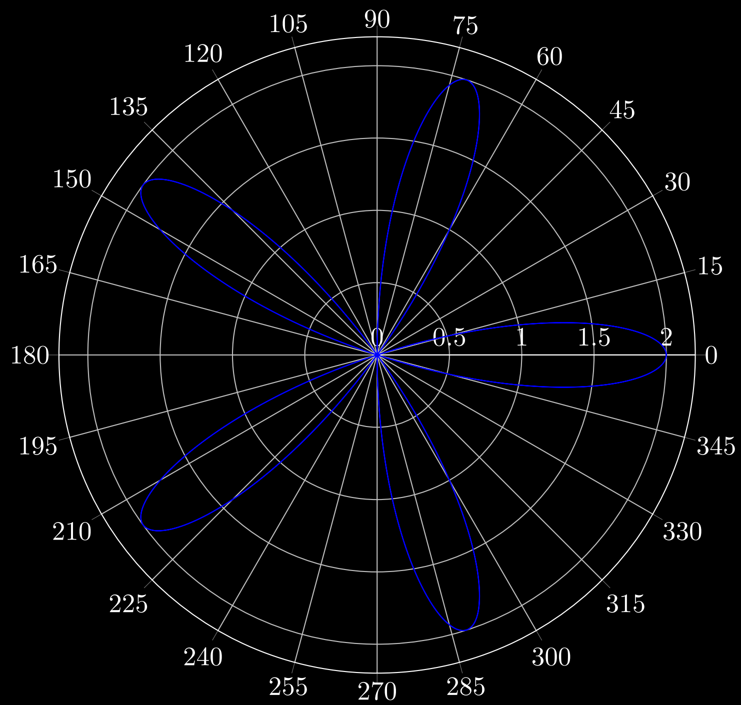
2.3 Problem 3

3. Suppose that you are asked to sketch the graph of $r = 2\cos(5\theta)$, without the help of a calculator. Discuss how you would proceed. Indicate what values of θ you would use to assist you in plotting significant points on the graph, and indicate what happens on the graph between successive significant points. Then sketch the graph of $r = 2\cos(5\theta)$.

I would proceed by plotting the significant points $\theta = 0, (\pi/2)/5, \pi/5, (3\pi/2)/5, 2\pi/5, \ldots$ etc. (incrementing by $\pi/10$) and keep track of the direction of the graph. Use the fact that $\pi/10$ is closer to 0 than known angle $\pi/6$, $2\pi/5$ is a little smaller than $2\pi/4 = \pi/2$, etc.

We could also use what we know from problem 2, that we need only plot from 0 to π for odd-n leaved roses $r = \cos(n\theta), r = \sin(n\theta)$.

θ	0	$\pi/10$	$\pi/5$	$3\pi/10$	$2\pi/5$	$\pi/2$	$6\pi/10 = 3\pi/5$	$7\pi/10$	$4\pi/5$	$9\pi/10$	π
f(θ) $= 2\cos(5\theta)$	2	0	-2	0	2	0	-2	0	2	0	-2



2.4 Problem 4

4. Consider the lemniscate $r^2 = 4\sin(2\theta)$, for $0 \le \theta \le 2\pi$.

(a) Sketch the lemniscate.

(b) For what values of θ in $[0, 2\pi]$ is there no real value of r? Indicate these values of θ on the graph in part (a).

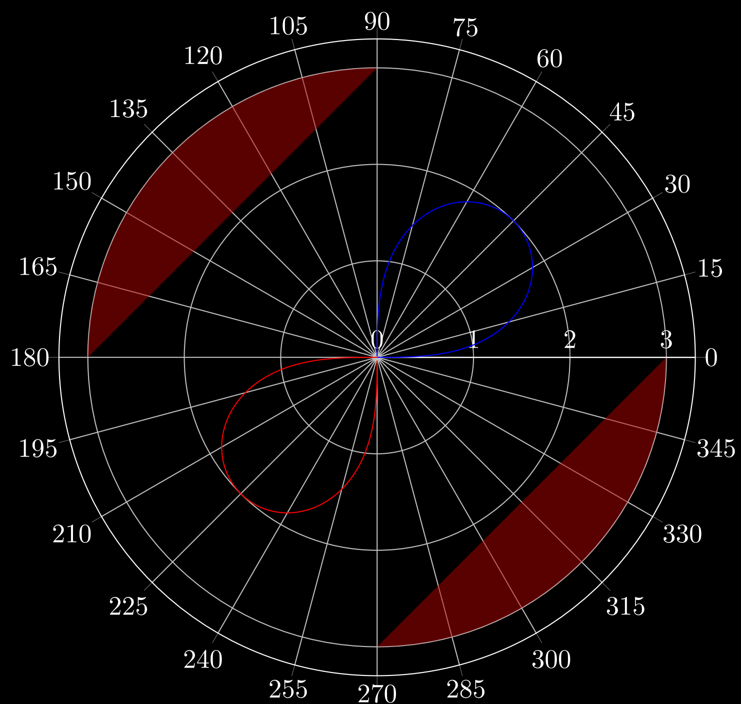
(c) Find $\int_0^{2\pi} \frac{1}{2}(4\sin(2\theta))d\theta$. Evidently the value of the integral is not the area A of the region enclosed by the lemniscate. Why? What is wrong with the given integral?

(d) Find the area A enclosed by the lemniscate. (Hint: Choose limits of integration judiciously.)

(a) Sketch the lemniscate.

To sketch the lemniscate, we consider significant points:

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	$\pi/2$	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$r : r^2 = 4\sin(2\theta)$	0	2	0	dne ($4\sin(2\theta) = -4$)	0	2	0	dne ($4\sin(2\theta) = -4$)	0



(b) For what values of θ in $[0, 2\pi]$ is there no real value of r? Indicate these values of θ on the graph in part (a).

If $\theta \in (\pi/2, \pi) \cup (3\pi/2, 2\pi)$, then $4\sin(2\theta) < 0$, so there's no real r with $r^2 = 4\sin(2\theta)$ since $r^2 \ge 0$ for any real number r. These values of θ correspond to the quadrants $\pi/2 < \theta < \pi$, $3\pi/2 < \theta < 2\pi$ with red in them.

(c) Find $\int_0^{2\pi} \frac{1}{2}(4\sin(2\theta))d\theta$. Evidently the value of the integral is not the area A of the region enclosed by the lemniscate. Why? What is wrong with the given integral?

The integral is $\int_0^{2\pi} \frac{1}{2}(4\sin(2\theta))d\theta = 2[-\frac{1}{2}\cos(2\theta)]_0^{2\pi} = (-1)-(-1) = 0$. This is incorrect since our integrand $r^2 = (f(\theta))^2 = 4\sin(2\theta)$ ignores the fact that $r^2 > 0$. Integrating $4\sin(2\theta)$ from 0 to 2π will allow negative areas, which we must avoid.

(d) Find the area A enclosed by the lemniscate. (Hint: Choose limits of integration judiciously.)

We need to adjust our bounds from $0 \leq \theta \leq 2\pi$ to $0 \leq \theta \leq \pi/2$ and $\pi \leq \theta \leq 3\pi/2$ to avoid negative values for the integrand $\frac{(f(\theta))^2}{2}$ of $A = \int_0^{2\pi} \frac{r^2}{2} d\theta$. Also, the graph is symmetric about the origin so $\int_0^{\pi/2} r^2/2 d\theta = \int_\pi^{3\pi/2} r^2/2 d\theta$. Then we have $A = \int_0^{\pi/2} 4\sin(2\theta)d\theta + \int_\pi^{3\pi/2} \frac{1}{2}(4\sin(2\theta))d\theta = 2\int_0^{\pi/2} \frac{1}{2}(4\sin(2\theta))d\theta = [-2\cos(2\theta)]_0^{\pi/2} = (-2(\cos(\pi/2)) - (-2\cos(0)) = (-2\cdot -1)+2 = 4$. Thus $A = 4$.

2.5 Problem 5

5. Write down a formula in polar coordinates for a function whose graph has the given symmetry, and draw the graph of the function.

(a) symmetry with respect to only the origin

(b) symmetry with respect to only the x axis

(c) symmetry with respect to only the y axis

To begin this problem we should also note the formal conditions for a polar function to be symmetric:

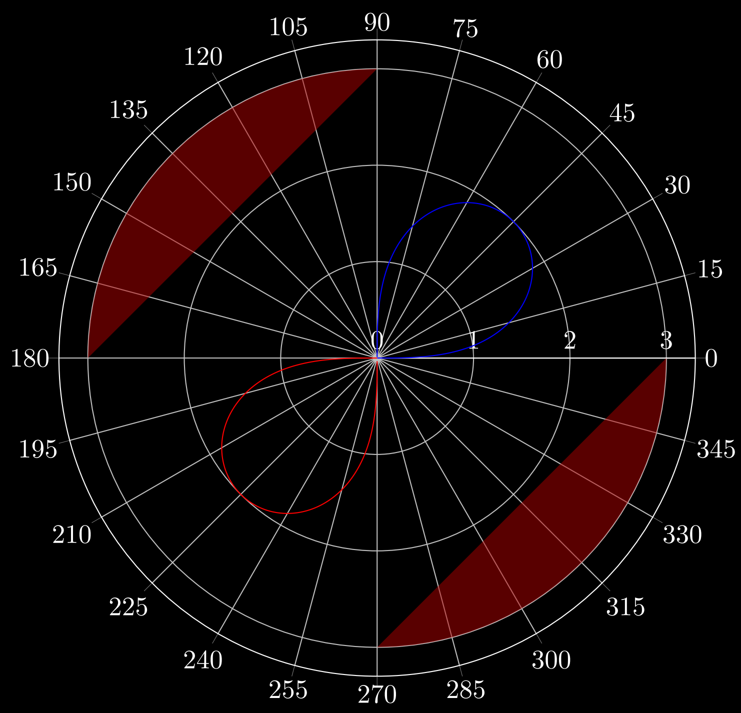
Formally, symmetry about the origin means if (r,θ) satisfies the equation (i.e. is on the graph), then $(-r,\theta)$ or $(r,\pi+\theta)$ also satisfies it.

Symmetry about the x axis means if (r,θ) satisfies the equation (i.e. is on the graph), then $(r,-\theta)$ or $(-r,\pi-\theta)$ is on the graph.

Lastly, symmetry about the origin means if (r,θ) satisfies the equation (i.e. is on the graph), then $(-r,-\theta)$ or $(r,\pi-\theta)$ also satisfies it.

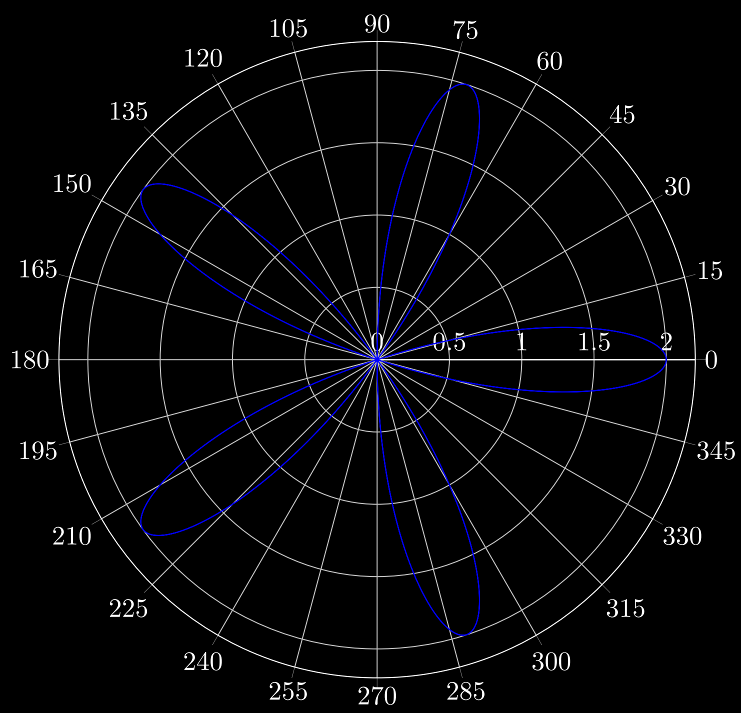
(a) symmetry with respect to only the origin

Notice the graph from Problem 4 works: $r^2 = 4\sin(2\theta)$. Note that symmetry with respect to only the origin means it is NOT symmetric with respect to the x or y axes alone.



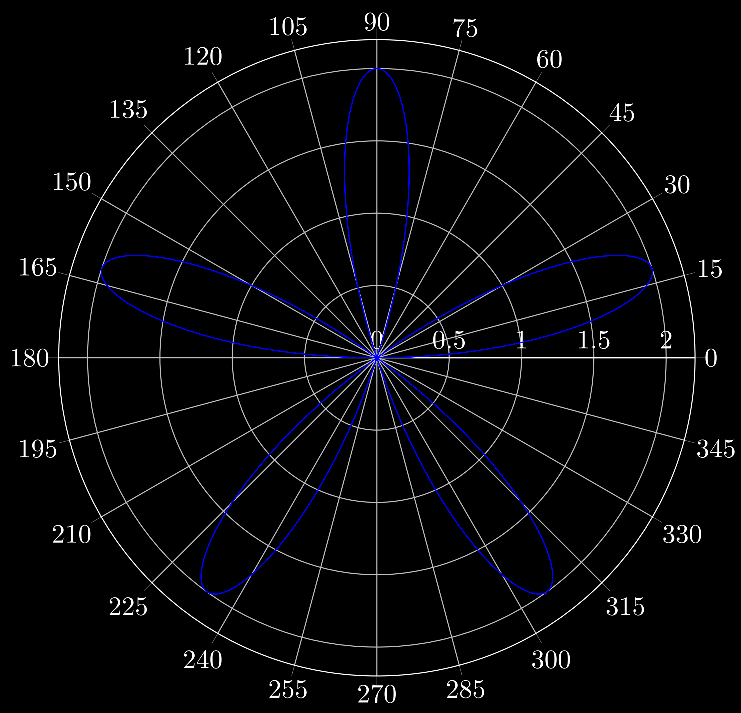
(b) symmetry with respect to only the x axis

The graph of $r = 2\cos(5\theta)$ from Problem 2 (section 2.3 in this file) works:



(c) symmetry with respect to only the y axis

Similar to (b), the graph of $r = 2\sin(5\theta)$ works:



90
75
60
45
30
15
0
345
330
315
300
285
270
255
240
225
210
195
180
165
150
135
120
105
0
0.5
1
1.5
2